

# NAG Library Function Document

## nag\_zhgeqz (f08xsc)

### 1 Purpose

nag\_zhgeqz (f08xsc) implements the  $QZ$  method for finding generalized eigenvalues of the complex matrix pair  $(A, B)$  of order  $n$ , which is in the generalized upper Hessenberg form.

### 2 Specification

```
#include <nag.h>
#include <nagf08.h>

void nag_zhgeqz (Nag_OrderType order, Nag_JobType job,
                Nag_ComputeQType compq, Nag_ComputeZType compz, Integer n, Integer ilo,
                Integer ihi, Complex a[], Integer pda, Complex b[], Integer pdb,
                Complex alpha[], Complex beta[], Complex q[], Integer pdq, Complex z[],
                Integer pdz, NagError *fail)
```

### 3 Description

nag\_zhgeqz (f08xsc) implements a single-shift version of the  $QZ$  method for finding the generalized eigenvalues of the complex matrix pair  $(A, B)$  which is in the generalized upper Hessenberg form. If the matrix pair  $(A, B)$  is not in the generalized upper Hessenberg form, then the function nag\_zgghrd (f08wsc) should be called before invoking nag\_zhgeqz (f08xsc).

This problem is mathematically equivalent to solving the matrix equation

$$\det(A - \lambda B) = 0.$$

Note that, to avoid underflow, overflow and other arithmetic problems, the generalized eigenvalues  $\lambda_j$  are never computed explicitly by this function but defined as ratios between two computed values,  $\alpha_j$  and  $\beta_j$ :

$$\lambda_j = \alpha_j / \beta_j.$$

The arguments  $\alpha_j$ , in general, are finite complex values and  $\beta_j$  are finite real non-negative values.

If desired, the matrix pair  $(A, B)$  may be reduced to generalized Schur form. That is, the transformed matrices  $A$  and  $B$  are upper triangular and the diagonal values of  $A$  and  $B$  provide  $\alpha$  and  $\beta$ .

The argument **job** specifies two options. If **job** = Nag\_Schur then the matrix pair  $(A, B)$  is simultaneously reduced to Schur form by applying one unitary transformation (usually called  $Q$ ) on the left and another (usually called  $Z$ ) on the right. That is,

$$\begin{aligned} A &\leftarrow Q^H A Z \\ B &\leftarrow Q^H B Z \end{aligned}$$

If **job** = Nag\_EigVals, then at each iteration the same transformations are computed but they are only applied to those parts of  $A$  and  $B$  which are needed to compute  $\alpha$  and  $\beta$ . This option could be used if generalized eigenvalues are required but not generalized eigenvectors.

If **job** = Nag\_Schur and **compq** = Nag\_AccumulateQ or Nag\_InitQ, and **compz** = Nag\_AccumulateZ or Nag\_InitZ, then the unitary transformations used to reduce the pair  $(A, B)$  are accumulated into the input arrays **q** and **z**. If generalized eigenvectors are required then **job** must be set to **job** = Nag\_Schur and if left (right) generalized eigenvectors are to be computed then **compq** (**compz**) must be set to **compq** = Nag\_AccumulateQ or Nag\_InitQ rather than **compq** = Nag\_NotQ.

If **compq** = Nag\_InitQ, then eigenvectors are accumulated on the identity matrix and on exit the array **q** contains the left eigenvector matrix  $Q$ . However, if **compq** = Nag\_AccumulateQ then the

transformations are accumulated in the user-supplied matrix  $Q_0$  in array **q** on entry and thus on exit **q** contains the matrix product  $QQ_0$ . A similar convention is used for **compz**.

## 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Moler C B and Stewart G W (1973) An algorithm for generalized matrix eigenproblems *SIAM J. Numer. Anal.* **10** 241–256

Stewart G W and Sun J-G (1990) *Matrix Perturbation Theory* Academic Press, London

## 5 Arguments

1: **order** – Nag\_OrderType *Input*

*On entry:* the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag\_RowMajor. See Section 2.3.1.3 in How to Use the NAG Library and its Documentation for a more detailed explanation of the use of this argument.

*Constraint:* **order** = Nag\_RowMajor or Nag\_ColMajor.

2: **job** – Nag\_JobType *Input*

*On entry:* specifies the operations to be performed on  $(A, B)$ .

**job** = Nag\_EigVals

The matrix pair  $(A, B)$  on exit might not be in the generalized Schur form.

**job** = Nag\_Schur

The matrix pair  $(A, B)$  on exit will be in the generalized Schur form.

*Constraint:* **job** = Nag\_EigVals or Nag\_Schur.

3: **compq** – Nag\_ComputeQType *Input*

*On entry:* specifies the operations to be performed on  $Q$ :

**compq** = Nag\_NotQ

The array **q** is unchanged.

**compq** = Nag\_AccumulateQ

The left transformation  $Q$  is accumulated on the array **q**.

**compq** = Nag\_InitQ

The array **q** is initialized to the identity matrix before the left transformation  $Q$  is accumulated in **q**.

*Constraint:* **compq** = Nag\_NotQ, Nag\_AccumulateQ or Nag\_InitQ.

4: **compz** – Nag\_ComputeZType *Input*

*On entry:* specifies the operations to be performed on  $Z$ .

**compz** = Nag\_NotZ

The array **z** is unchanged.

**compz** = Nag\_AccumulateZ

The right transformation  $Z$  is accumulated on the array **z**.

**compz** = Nag\_InitZ

The array **z** is initialized to the identity matrix before the right transformation  $Z$  is accumulated in **z**.

*Constraint:* **compz** = Nag\_NotZ, Nag\_AccumulateZ or Nag\_InitZ.

5: **n** – Integer *Input*

*On entry:*  $n$ , the order of the matrices  $A$ ,  $B$ ,  $Q$  and  $Z$ .

*Constraint:*  $n \geq 0$ .

6: **ilo** – Integer *Input*

7: **ihi** – Integer *Input*

*On entry:* the indices  $i_{lo}$  and  $i_{hi}$ , respectively which define the upper triangular parts of  $A$ . The submatrices  $A(1 : i_{lo} - 1, 1 : i_{lo} - 1)$  and  $A(i_{hi} + 1 : n, i_{hi} + 1 : n)$  are then upper triangular. These arguments are provided by nag\_zggbal (f08wvc) if the matrix pair was previously balanced; otherwise, **ilo** = 1 and **ihi** = **n**.

*Constraints:*

if  $n > 0$ ,  $1 \leq \mathbf{ilo} \leq \mathbf{ihi} \leq n$ ;  
if  $n = 0$ , **ilo** = 1 and **ihi** = 0.

8: **a**[*dim*] – Complex *Input/Output*

**Note:** the dimension, *dim*, of the array **a** must be at least  $\max(1, \mathbf{pda} \times n)$ .

The ( $i, j$ )th element of the matrix  $A$  is stored in

**a**[( $j - 1$ )  $\times$  **pda** +  $i - 1$ ] when **order** = Nag\_ColMajor;  
**a**[( $i - 1$ )  $\times$  **pda** +  $j - 1$ ] when **order** = Nag\_RowMajor.

*On entry:* the  $n$  by  $n$  upper Hessenberg matrix  $A$ . The elements below the first subdiagonal must be set to zero.

*On exit:* if **job** = Nag\_Schur, the matrix pair  $(A, B)$  will be simultaneously reduced to generalized Schur form.

If **job** = Nag\_EigVals, the 1 by 1 and 2 by 2 diagonal blocks of the matrix pair  $(A, B)$  will give generalized eigenvalues but the remaining elements will be irrelevant.

9: **pda** – Integer *Input*

*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **a**.

*Constraint:* **pda**  $\geq \max(1, n)$ .

10: **b**[*dim*] – Complex *Input/Output*

**Note:** the dimension, *dim*, of the array **b** must be at least  $\max(1, \mathbf{pdb} \times n)$ .

The ( $i, j$ )th element of the matrix  $B$  is stored in

**b**[( $j - 1$ )  $\times$  **pdb** +  $i - 1$ ] when **order** = Nag\_ColMajor;  
**b**[( $i - 1$ )  $\times$  **pdb** +  $j - 1$ ] when **order** = Nag\_RowMajor.

*On entry:* the  $n$  by  $n$  upper triangular matrix  $B$ . The elements below the diagonal must be zero.

*On exit:* if **job** = Nag\_Schur, the matrix pair  $(A, B)$  will be simultaneously reduced to generalized Schur form.

If **job** = Nag\_EigVals, the 1 by 1 and 2 by 2 diagonal blocks of the matrix pair  $(A, B)$  will give generalized eigenvalues but the remaining elements will be irrelevant.

- 11: **pdb** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **b**.  
*Constraint:*  $\mathbf{pdb} \geq \max(1, \mathbf{n})$ .
- 12: **alpha[n]** – Complex *Output*  
*On exit:*  $\alpha_j$ , for  $j = 1, 2, \dots, n$ .
- 13: **beta[n]** – Complex *Output*  
*On exit:*  $\beta_j$ , for  $j = 1, 2, \dots, n$ .
- 14: **q[dim]** – Complex *Input/Output*  
**Note:** the dimension, *dim*, of the array **q** must be at least  
 $\max(1, \mathbf{pdq} \times \mathbf{n})$  when **compq** = Nag\_AccumulateQ or Nag\_InitQ;  
 1 when **compq** = Nag\_NotQ.  
 The (*i*, *j*)th element of the matrix *Q* is stored in  
 $\mathbf{q}[(j-1) \times \mathbf{pdq} + i - 1]$  when **order** = Nag\_ColMajor;  
 $\mathbf{q}[(i-1) \times \mathbf{pdq} + j - 1]$  when **order** = Nag\_RowMajor.  
*On entry:* if **compq** = Nag\_AccumulateQ, the matrix  $Q_0$ . The matrix  $Q_0$  is usually the matrix *Q* returned by nag\_zgghrd (f08wsc).  
 If **compq** = Nag\_NotQ, **q** is not referenced.  
*On exit:* if **compq** = Nag\_AccumulateQ, **q** contains the matrix product  $QQ_0$ .  
 If **compq** = Nag\_InitQ, **q** contains the transformation matrix *Q*.
- 15: **pdq** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **q**.  
*Constraints:*  
 if **order** = Nag\_ColMajor,  
     if **compq** = Nag\_AccumulateQ or Nag\_InitQ,  $\mathbf{pdq} \geq \mathbf{n}$ ;  
     if **compq** = Nag\_NotQ,  $\mathbf{pdq} \geq 1$ .;  
 if **order** = Nag\_RowMajor,  
     if **compq** = Nag\_AccumulateQ or Nag\_InitQ,  $\mathbf{pdq} \geq \max(1, \mathbf{n})$ ;  
     if **compq** = Nag\_NotQ,  $\mathbf{pdq} \geq 1$ ..
- 16: **z[dim]** – Complex *Input/Output*  
**Note:** the dimension, *dim*, of the array **z** must be at least  
 $\max(1, \mathbf{pdz} \times \mathbf{n})$  when **compz** = Nag\_AccumulateZ or Nag\_InitZ;  
 1 when **compz** = Nag\_NotZ.  
 The (*i*, *j*)th element of the matrix *Z* is stored in  
 $\mathbf{z}[(j-1) \times \mathbf{pdz} + i - 1]$  when **order** = Nag\_ColMajor;  
 $\mathbf{z}[(i-1) \times \mathbf{pdz} + j - 1]$  when **order** = Nag\_RowMajor.  
*On entry:* if **compz** = Nag\_AccumulateZ, the matrix  $Z_0$ . The matrix  $Z_0$  is usually the matrix *Z* returned by nag\_zgghrd (f08wsc).  
 If **compz** = Nag\_NotZ, **z** is not referenced.  
*On exit:* if **compz** = Nag\_AccumulateZ, **z** contains the matrix product  $ZZ_0$ .

If **compz** = Nag\_InitZ, **z** contains the transformation matrix *Z*.

17: **pdz** – Integer *Input*

*On entry*: the stride separating row or column elements (depending on the value of **order**) in the array **z**.

*Constraints*:

if **order** = Nag\_ColMajor,  
     if **compz** = Nag\_AccumulateZ or Nag\_InitZ, **pdz** ≥ **n**;  
     if **compz** = Nag\_NotZ, **pdz** ≥ 1.;  
 if **order** = Nag\_RowMajor,  
     if **compz** = Nag\_AccumulateZ or Nag\_InitZ, **pdz** ≥ max(1, **n**);  
     if **compz** = Nag\_NotZ, **pdz** ≥ 1..

18: **fail** – NagError \* *Input/Output*

The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

### NE\_BAD\_PARAM

On entry, argument *⟨value⟩* had an illegal value.

### NE\_ENUM\_INT\_2

On entry, **compq** = *⟨value⟩*, **pdq** = *⟨value⟩*, **n** = *⟨value⟩*.

Constraint: if **compq** = Nag\_AccumulateQ or Nag\_InitQ, **pdq** ≥ max(1, **n**);  
 if **compq** = Nag\_NotQ, **pdq** ≥ 1.

On entry, **compq** = *⟨value⟩*, **pdq** = *⟨value⟩* and **n** = *⟨value⟩*.

Constraint: if **compq** = Nag\_AccumulateQ or Nag\_InitQ, **pdq** ≥ **n**;  
 if **compq** = Nag\_NotQ, **pdq** ≥ 1.

On entry, **compz** = *⟨value⟩*, **pdz** = *⟨value⟩*, **n** = *⟨value⟩*.

Constraint: if **compz** = Nag\_AccumulateZ or Nag\_InitZ, **pdz** ≥ max(1, **n**);  
 if **compz** = Nag\_NotZ, **pdz** ≥ 1.

On entry, **compz** = *⟨value⟩*, **pdz** = *⟨value⟩* and **n** = *⟨value⟩*.

Constraint: if **compz** = Nag\_AccumulateZ or Nag\_InitZ, **pdz** ≥ **n**;  
 if **compz** = Nag\_NotZ, **pdz** ≥ 1.

### NE\_INT

On entry, **n** = *⟨value⟩*.

Constraint: **n** ≥ 0.

On entry, **pda** = *⟨value⟩*.

Constraint: **pda** > 0.

On entry, **pdb** = *⟨value⟩*.

Constraint: **pdb** > 0.

On entry, **pdq** = *⟨value⟩*.

Constraint: **pdq** > 0.

On entry, **pdz** =  $\langle value \rangle$ .  
 Constraint: **pdz** > 0.

### NE\_INT\_2

On entry, **pda** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pda**  $\geq$  max(1, **n**).

On entry, **pdb** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pdb**  $\geq$  max(1, **n**).

### NE\_INT\_3

On entry, **n** =  $\langle value \rangle$ , **ilo** =  $\langle value \rangle$  and **ihi** =  $\langle value \rangle$ .  
 Constraint: if **n** > 0,  $1 \leq \mathbf{ilo} \leq \mathbf{ihi} \leq \mathbf{n}$ ;  
 if **n** = 0, **ilo** = 1 and **ihi** = 0.

### NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.  
 See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

An unexpected Library error has occurred.

### NE\_ITERATION\_QZ

The *QZ* iteration did not converge and the matrix pair  $(A, B)$  is not in the generalized Schur form. The computed  $\alpha_i$  and  $\beta_i$  should be correct for  $i = \langle value \rangle, \dots, \langle value \rangle$ .

### NE\_NO\_LICENCE

Your licence key may have expired or may not have been installed correctly.  
 See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

### NE\_SCHUR

The computation of shifts failed and the matrix pair  $(A, B)$  is not in the generalized Schur form. The computed  $\alpha_i$  and  $\beta_i$  should be correct for  $i = \langle value \rangle, \dots, \langle value \rangle$ .

## 7 Accuracy

Please consult Section 4.11 of the LAPACK Users' Guide (see Anderson *et al.* (1999)) and Chapter 6 of Stewart and Sun (1990), for more information.

## 8 Parallelism and Performance

nag\_zhgeqz (f08xsc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the x06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this function. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

nag\_zhgeqz (f08xsc) is the fifth step in the solution of the complex generalized eigenvalue problem and is called after nag\_zgghrd (f08wsc).

The number of floating-point operations taken by this function is proportional to  $n^3$ .

The real analogue of this function is nag\_dhgeqz (f08xec).

## 10 Example

This example computes the  $\alpha$  and  $\beta$  arguments, which defines the generalized eigenvalues, of the matrix pair  $(A, B)$  given by

$$A = \begin{pmatrix} 1.0 + 3.0i & 1.0 + 4.0i & 1.0 + 5.0i & 1.0 + 6.0i \\ 2.0 + 2.0i & 4.0 + 3.0i & 8.0 + 4.0i & 16.0 + 5.0i \\ 3.0 + 1.0i & 9.0 + 2.0i & 27.0 + 3.0i & 81.0 + 4.0i \\ 4.0 + 0.0i & 16.0 + 1.0i & 64.0 + 2.0i & 256.0 + 3.0i \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1.0 + 0.0i & 2.0 + 1.0i & 3.0 + 2.0i & 4.0 + 3.0i \\ 1.0 + 1.0i & 4.0 + 2.0i & 9.0 + 3.0i & 16.0 + 4.0i \\ 1.0 + 2.0i & 8.0 + 3.0i & 27.0 + 4.0i & 64.0 + 5.0i \\ 1.0 + 3.0i & 16.0 + 4.0i & 81.0 + 5.0i & 256.0 + 6.0i \end{pmatrix}.$$

This requires calls to five functions: nag\_zggbal (f08wvc) to balance the matrix, nag\_zgeqrf (f08asc) to perform the  $QR$  factorization of  $B$ , nag\_zunmqr (f08auc) to apply  $Q$  to  $A$ , nag\_zgghrd (f08wsc) to reduce the matrix pair to the generalized Hessenberg form and nag\_zhgeqz (f08xsc) to compute the eigenvalues using the  $QZ$  algorithm.

### 10.1 Program Text

```

/* nag_zhgeqz (f08xsc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 */

#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <naga02.h>
#include <nagm01.h>
#include <nagf08.h>
#include <nagx04.h>

#ifdef __cplusplus
extern "C"
{
#endif
    static Integer NAG_CALL compare(const Nag_Pointer a, const Nag_Pointer b);
    static Integer sort_values (Integer n, Complex alpha[], Complex beta[],
                               Complex e[], size_t rank[], double emod[]);
#ifdef __cplusplus
}
#endif

int main(void)
{
    /* Scalars */
    Integer      i, ihi, ilo, irows, j, n, pda, pdb, ninf;
    Integer      exit_status = 0;

    /* Arrays */
    Complex      *a = 0, *alpha = 0, *b = 0, *beta = 0, *q = 0, *tau = 0;
    Complex      *z = 0, *e = 0;
    double       *lscale = 0, *rscale = 0, *emod = 0;
    size_t       *rank = 0;

    /* Nag Types */

```

```

NagError      fail;
Nag_OrderType order;

#ifdef NAG_COLUMN_MAJOR
#define A(I, J) a[(J-1)*pda + I - 1]
#define B(I, J) b[(J-1)*pdb + I - 1]
    order = Nag_ColMajor;
#else
#define A(I, J) a[(I-1)*pda + J - 1]
#define B(I, J) b[(I-1)*pdb + J - 1]
    order = Nag_RowMajor;
#endif

    INIT_FAIL(fail);

    printf("nag_zhgeqz (f08xsc) Example Program Results\n\n");
    /* Skip heading in data file */
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif
#ifdef _WIN32
    scanf_s("%" NAG_IFMT "%*[\n] ", &n);
#else
    scanf("%" NAG_IFMT "%*[\n] ", &n);
#endif
    pda = n;
    pdb = n;

    /* Allocate memory */
    if (!(a      = NAG_ALLOC(n * n, Complex)) ||
        !(b      = NAG_ALLOC(n * n, Complex)) ||
        !(alpha  = NAG_ALLOC(n, Complex)) ||
        !(beta   = NAG_ALLOC(n, Complex)) ||
        !(q      = NAG_ALLOC(1 * 1, Complex)) ||
        !(tau    = NAG_ALLOC(n, Complex)) ||
        !(lscale = NAG_ALLOC(n, double)) ||
        !(rscale = NAG_ALLOC(n, double)) ||
        !(rank   = NAG_ALLOC(n, size_t)) ||
        !(emod   = NAG_ALLOC(n, double)) ||
        !(e      = NAG_ALLOC(n, Complex)) ||
        !(z      = NAG_ALLOC(1 * 1, Complex)))
    {
        printf("Allocation failure\n");
        exit_status = -1;
        goto END;
    }

    /* READ matrix A from data file */
    for (i = 1; i <= n; ++i) {
        for (j = 1; j <= n; ++j)
#ifdef _WIN32
            scanf_s(" ( %lf, %lf ) ", &A(i, j).re, &A(i, j).im);
#else
            scanf(" ( %lf, %lf ) ", &A(i, j).re, &A(i, j).im);
#endif
    }
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif

    /* READ matrix B from data file */
    for (i = 1; i <= n; ++i) {
        for (j = 1; j <= n; ++j)
#ifdef _WIN32
            scanf_s(" ( %lf, %lf ) ", &B(i, j).re, &B(i, j).im);
#else
            scanf(" ( %lf, %lf ) ", &B(i, j).re, &B(i, j).im);
#endif
    }

```



```

#endif
}
#ifdef _WIN32
scanf_s("%*[\n] ");
#else
scanf("%*[\n] ");
#endif
/* Balance matrix pair (A,B) */
/* nag_zggbal (f08wvc).
 * Balance a pair of complex general matrices
 */
nag_zggbal(order, Nag_DoBoth, n, a, pda, b, pdb, &ilo, &ihi, lscale,
           rscale, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zggbal (f08wvc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Matrix A after balancing */
/* nag_gen_complx_mat_print_comp (x04dbc).
 * Print complex general matrix (comprehensive)
 */
fflush(stdout);
nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n,
                              n, a, pda, Nag_BracketForm, "%7.4f",
                              "Matrix A after balancing",
                              Nag_IntegerLabels, 0, Nag_IntegerLabels, 0,
                              80, 0, 0, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
          fail.message);
    exit_status = 2;
    goto END;
}
printf("\n");

/* Matrix B after balancing */
/* nag_gen_complx_mat_print_comp (x04dbc), see above. */
fflush(stdout);
nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n,
                              n, b, pdb, Nag_BracketForm, "%7.4f",
                              "Matrix B after balancing",
                              Nag_IntegerLabels, 0, Nag_IntegerLabels, 0,
                              80, 0, 0, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
          fail.message);
    exit_status = 3;
    goto END;
}
printf("\n");

/* Reduce B to triangular form using QR */
irows = ihi + 1 - ilo;
/* nag_zgeqrf (f08asc).
 * QR factorization of complex general rectangular matrix
 */
nag_zgeqrf(order, irows, irows, &B(ilo, ilo), pdb, tau, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zgeqrf (f08asc).\n%s\n", fail.message);
    exit_status = 4;
    goto END;
}

/* Apply the orthogonal transformation to matrix A */
/* nag_zunmqr (f08auc).
 * Apply unitary transformation determined by nag_zgeqrf
 * (f08asc) or nag_zgeqpf (f08bsc)
 */
nag_zunmqr(order, Nag_LeftSide, Nag_ConjTrans, irows, irows, irows,

```

```

        &B(ilo, ilo), pdb, tau, &A(ilo, ilo), pda, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zunmqr (f08auc).\n%s\n", fail.message);
    exit_status = 5;
    goto END;
}

/* Compute the generalized Hessenberg form of (A,B) */
/* nag_zgghd3 (f08wtc).
 * Unitary reduction of a pair of complex general matrices
 * to generalized upper Hessenberg form
 */
nag_zgghd3(order, Nag_NotQ, Nag_NotZ, irows, 1, irows, &A(ilo, ilo),
           pda, &B(ilo, ilo), pdb, q, 1, z, 1, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zgghd3 (f08wtc).\n%s\n", fail.message);
    exit_status = 6;
    goto END;
}

/* Matrix A in generalized Hessenberg form */
/* nag_gen_complx_mat_print_comp (x04dbc), see above. */
fflush(stdout);
nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n,
                              n, a, pda, Nag_BracketForm, "%7.3f",
                              "Matrix A in Hessenberg form",
                              Nag_IntegerLabels, 0, Nag_IntegerLabels, 0,
                              80, 0, 0, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
          fail.message);
    exit_status = 7;
    goto END;
}
printf("\n");
/* Matrix B in generalized Hessenberg form */
/* nag_gen_complx_mat_print_comp (x04dbc), see above. */
fflush(stdout);
nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, n,
                              n, b, pdb, Nag_BracketForm, "%7.3f",
                              "Matrix B is triangular",
                              Nag_IntegerLabels, 0, Nag_IntegerLabels, 0,
                              80, 0, 0, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
          fail.message);
    exit_status = 8;
    goto END;
}

/* Compute the generalized Schur form */
/* nag_zhgeqz (f08xsc).
 * Eigenvalues and generalized Schur factorization of
 * complex generalized upper Hessenberg form reduced from a
 * pair of complex general matrices
 */
nag_zhgeqz(order, Nag_EigVals, Nag_NotQ, Nag_NotZ, n, ilo, ihi, a,
           pda, b, pdb, alpha, beta, q, 1, z, 1, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zhgeqz (f08xsc).\n%s\n", fail.message);
    exit_status = 9;
    goto END;
}

/* Print the generalized eigenvalues */
printf("\n Generalized eigenvalues\n");
ninf = 0;
for (i = 0; i < n; ++i) {
    if (beta[i].re == 0.0) {
        ninf = ninf + 1;
        printf(" %4" NAG_IFMT "      Infinite eigenvalue\n", i + 1);
    }
}

```

```

    }
}
if (ninf==0) {
    /* Sort values by decreasing modulus and store in e[] */
    exit_status=sort_values (n, alpha, beta, e, rank, emod);
    for (i = 0; i < n; ++i) {
        printf(" %4" NAG_IFMT "      (%7.3f,%7.3f)\n", i + 1, e[i].re, e[i].im);
    }
}
END:
NAG_FREE(a);
NAG_FREE(alpha);
NAG_FREE(b);
NAG_FREE(beta);
NAG_FREE(lscale);
NAG_FREE(q);
NAG_FREE(rank);
NAG_FREE(rscale);
NAG_FREE(tau);
NAG_FREE(e);
NAG_FREE(emod);
NAG_FREE(z);

return exit_status;
}
static Integer sort_values (Integer n, Complex alpha[], Complex beta[],
                           Complex e[], size_t rank[], double emod[])
{
    Integer      i, exit_status = 0;
    NagError     fail;

    INIT_FAIL(fail);

    for (i = 0; i < n; ++i) {
        /* nag_complex_divide (a02cdc): Quotient of two complex numbers;
         * nag_complex_abs (a02ddc): Moduli of complex number.
         */
        e[i] = nag_complex_divide(alpha[i], beta[i]);
        emod[i] = nag_complex_abs(e[i]);
    }
    /* Rank sort eigenvalues by absolute values using
     * nag_rank_sort (m0ldsc).
     */
    nag_rank_sort((Pointer) emod, (size_t) n, (ptrdiff_t) (sizeof(double)),
                  compare, Nag_Descending, rank, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_rank_sort (m0ldsc).\n%s\n", fail.message);
        exit_status = 10;
        goto END;
    }
    /* Turn ranks into indices using nag_make_indices (m0lzac). */
    nag_make_indices(rank, (size_t) n, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_make_indices (m0lzac).\n%s\n", fail.message);
        exit_status = 11;
        goto END;
    }
    /* Sort eigenvalues using nag_reorder_vector (m0lesc). */
    nag_reorder_vector((Pointer) e, (size_t) n, sizeof(Complex),
                       (ptrdiff_t) sizeof(Complex), rank, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_reorder_vector (m0lesc).\n%s\n", fail.message);
        exit_status = 12;
        goto END;
    }
}
END:
return exit_status;
}

```

```
static Integer NAG_CALL compare(const Nag_Pointer a, const Nag_Pointer b)
{
  double x = *((const double *) a) - *((const double *) b);
  return (x < 0.0 ? -1 : (x == 0.0 ? 0 : 1));
}
```

## 10.2 Program Data

```
nag_zhgeqz (f08xsc) Example Program Data
      4                                     :Value of N
( 1.00, 3.00) ( 1.00, 4.00) ( 1.00, 5.00) ( 1.00, 6.00)
( 2.00, 2.00) ( 4.00, 3.00) ( 8.00, 4.00) (16.00, 5.00)
( 3.00, 1.00) ( 9.00, 2.00) (27.00, 3.00) (81.00, 4.00)
( 4.00, 0.00) (16.00, 1.00) (64.00, 2.00) (256.00, 3.00) :End of matrix A
( 1.00, 0.00) ( 2.00, 1.00) ( 3.00, 2.00) ( 4.00, 3.00)
( 1.00, 1.00) ( 4.00, 2.00) ( 9.00, 3.00) (16.00, 4.00)
( 1.00, 2.00) ( 8.00, 3.00) (27.00, 4.00) (64.00, 5.00)
( 1.00, 3.00) (16.00, 4.00) (81.00, 5.00) (256.00, 6.00) :End of matrix B
```

## 10.3 Program Results

```
nag_zhgeqz (f08xsc) Example Program Results
```

Matrix A after balancing

```

      1          2          3          4
1 ( 1.0000, 3.0000) ( 1.0000, 4.0000) ( 0.1000, 0.5000) ( 0.1000, 0.6000)
2 ( 2.0000, 2.0000) ( 4.0000, 3.0000) ( 0.8000, 0.4000) ( 1.6000, 0.5000)
3 ( 0.3000, 0.1000) ( 0.9000, 0.2000) ( 0.2700, 0.0300) ( 0.8100, 0.0400)
4 ( 0.4000, 0.0000) ( 1.6000, 0.1000) ( 0.6400, 0.0200) ( 2.5600, 0.0300)
```

Matrix B after balancing

```

      1          2          3          4
1 ( 1.0000, 0.0000) ( 2.0000, 1.0000) ( 0.3000, 0.2000) ( 0.4000, 0.3000)
2 ( 1.0000, 1.0000) ( 4.0000, 2.0000) ( 0.9000, 0.3000) ( 1.6000, 0.4000)
3 ( 0.1000, 0.2000) ( 0.8000, 0.3000) ( 0.2700, 0.0400) ( 0.6400, 0.0500)
4 ( 0.1000, 0.3000) ( 1.6000, 0.4000) ( 0.8100, 0.0500) ( 2.5600, 0.0600)
```

Matrix A in Hessenberg form

```

      1          2          3          4
1 (-2.868, -1.595) (-0.809, -0.328) (-4.900, -0.987) (-0.048, 1.163)
2 (-2.672, 0.595) (-0.790, 0.049) (-4.955, -0.163) (-0.439, -0.574)
3 ( 0.000, 0.000) (-0.098, -0.011) (-1.168, -0.137) (-1.756, -0.205)
4 ( 0.000, 0.000) ( 0.000, 0.000) ( 0.087, 0.004) ( 0.032, 0.001)
```

Matrix B is triangular

```

      1          2          3          4
1 (-1.775, 0.000) (-0.721, 0.043) (-5.021, 1.190) (-0.145, 0.726)
2 ( 0.000, 0.000) (-0.218, 0.035) (-2.541, -0.146) (-0.823, -0.418)
3 ( 0.000, 0.000) ( 0.000, 0.000) (-1.396, -0.163) (-1.747, -0.204)
4 ( 0.000, 0.000) ( 0.000, 0.000) ( 0.000, 0.000) (-0.100, -0.004)
```

Generalized eigenvalues

```

1 ( -0.635, 1.653)
2 ( 0.493, 0.910)
3 ( 0.458, -0.843)
4 ( 0.674, -0.050)
```

---